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A Note on the Local Stability of Translates of Radial Basis Functions

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We give sufficient conditions on a radial basis function that imply that the multi-integer translates of this radial function satisfy a local stability estimate in the uniform norm. © 1993 Academic Press, Inc.

In this note, we study the question of local stability of functions from radial basis function spaces. Specifically, let $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ be a continuous "radial basis function" and denote the composite function $\phi \circ \|\cdot\|\colon \mathbb{R}^s \to \mathbb{R}$ (where $\|\cdot\|$ is the Euclidean norm) also by ϕ. Approximations from spaces generated by translates of this function ϕ have been subject to a lively investigation in the recent past (see, e.g., Powell [2] for a comprehensive survey), and here, interest has frequently been focused on integer translates of ϕ. While one of the main interests in the literature has been a study of the existence and uniqueness of interpolants from spaces spanned by translates of radial basis functions, and the approximational efficacy thereof (see, for instance, Buhmann [1]), our goal in this note is to give sufficient conditions on ϕ that ensure that translates of ϕ along the multi-integers satisfy the local stability estimate

$$|d_l| \leqslant C \sup_{x-l\in K} \left| \sum_{j\in\mathbb{Z}^s} d_j \phi(x-j) \right| \qquad \text{for all} \quad l \in \mathbb{Z}^s. \tag{1}$$

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Here, K is a compact neighbourhood about zero, C is a positive constant that does not depend on l or on the d_j but does depend on K, and the $\{d_j\}_{j\in\mathbb{Z}^s}$ are required to be such that

$$\sum_{j\in\mathbb{Z}^s} |d_j\phi(x-j)| < \infty \qquad \text{for all} \quad x\in\mathbb{R}^s. \tag{2}$$

We note that the estimate (1) provides, in particular, a new proof of the fact that the coefficients of some of the fundamental functions for interpolation which were studied in Buhmann [1] decay at least as fast as the fundamental functions themselves. We also note that, if $|\phi(x)|$ decays fast enough at infinity for

$$\left\| \sum_{j\in\mathbb{Z}^s} d_j\phi(\cdot-j)\right\|_\infty := \sup_{x\in\mathbb{R}^s}\left|\sum_{j\in\mathbb{Z}^s} d_j\phi(x-j)\right|$$

to be finite for all sequences $\{d_j\}_{j\in\mathbb{Z}^s}$ that are such that

$$\|\{d_j\}_{j\in\mathbb{Z}^s}\|_\infty := \sup_{j\in\mathbb{Z}^s} |d_j| < \infty,$$

then (1) implies

$$\|\{d_j\}_{j\in\mathbb{Z}^s}\|_\infty \leqslant C\left\|\sum_{j\in\mathbb{Z}^s} d_j\phi(\cdot-j)\right\|_\infty \leqslant C'\,\|\{d_j\}_{j\in\mathbb{Z}^s}\|_\infty.$$

This is the standard stability estimate in the uniform norm.

We have the following result, which applies, for instance, when $\phi(r)$ is any odd power of r (which is a choice frequently discussed in the literature) and the dimension of the underlying space is odd too. Odd powers of the Euclidean norm are fundamental solutions of iterated Laplacian operators, which is a highly relevant fact in the analysis of radial basis function approximation methods (see, for instance, Powell [2]), but this is only true when the spatial dimension is odd, since in even dimensions expressions of the form $r^{2k}\log r$ are the fundamental solutions of the iterated Laplace operators.

THEOREM. *Suppose $\phi\colon \mathbb{R}_+\to\mathbb{R}$ is not identically zero and has the form*

$$\phi(r) = \sum_{l=0}^{(m-1)/2} a_l r^{m-2l}, \qquad r\geqslant 0,$$

where m is an odd positive integer. Let s be positive and odd too. Then the stability estimate (1) *holds.*

Proof. We remark first that in the one-dimensional case, any linear combination of integer translates $\phi(|\cdot - j|)$ for such a ϕ as is required in the statement of the theorem, is a piecewise polynomial, so the univariate spline theory (see, e.g., Schumaker [4]) provides the desired result. So let $s \geq 3$. The salient ingredient to the proof is identifying functionals that are dual to our radial function and its translates, in an integral representation, where the range of integration is a compact set, which will immediately lead to the estimate (1). To this end, let us assume without loss of generality that the ball about zero of radius 1/2 is contained in K. We note the identity which is useful in obtaining the aforementioned dual functionals

$$\int_{\|x\|=r} \phi(x-y)\,dx = H_s r^{s-1} \sum_{l=0}^{(m-1)/2} a_l \|y\|^{m-2l} \times \sum_{j=0}^{(s+m)/2-l-1} b_{l,j} \left(\frac{r}{\|y\|}\right)^{2j}, \qquad r < \|y\|, \tag{3}$$

where H_s and $\{b_{l,j}\}$ are real constants, and $H_s > 0$, $b_{l,0} \equiv b_0 > 0$, i.e., the $b_{l,0}$ do not depend on l and are positive. Before we prove (3), we will first show how (1) is derived from (3). Choose numbers $\{t_k\}_{k\in J} \subset \mathbb{R}$, $\{r_k\}_{k\in J} \subset (0, 1/2)$, where $J \subset \mathbb{Z}_+$ is a finite subset, such that the inequality in

$$\sum_{k\in J} t_k \int_{\|x\|=r_k} \phi(x)\,dx = \sum_{l=0}^{(m-1)/2} a_l \sum_{k\in J} t_k \int_{\|x\|=r_k} \|x\|^{m-2l}\,dx$$
$$= \frac{2\pi^{s/2}}{\Gamma(\frac{1}{2}s)} \sum_{l=0}^{(m-1)/2} a_l \sum_{k\in J} t_k r_k^{s-1+m-2l}\,dx \neq \phi(0) = 0$$

holds and such that it is true that

$$\sum_{k\in J} t_k r_k^{2j+s-1} = 0 \qquad \forall \quad 0 < j < (s+m)/2,$$
$$\sum_{k\in J} t_k r_k^{s-1} = \frac{1}{H_s b_0}. \tag{4}$$

Such numbers exist by the linear independence of the powers r_k^{j+s-1} if $J \subset \mathbb{Z}_+$ is large enough. If we now define the linear functional

$$Lf : f \to f(0) - \sum_{k\in J} t_k \int_{\|x\|=r_k} f(x)\,dx,$$

then for some nonzero constant $\tilde{C}$, we have, by (3) and (4),

$$L\left[\sum_{j\in\mathbb{Z}^s} d_j \phi(\cdot - j)\right] = \sum_{j\in\mathbb{Z}^s} d_j [L\phi(\cdot - j)] = \tilde{C} d_0, \tag{5}$$

where it is admissible to take the linear functional inside the sum because of the summability requirement (2). We note that (5) means that L and its integer translates are dual functionals to the integer translates of ϕ.

The estimate (1) follows from the fact that we have identified an integral representation for the functional that is dual to ϕ and its translates. Specifically, we estimate

$$\left| L\left[\sum_{j \in \mathbb{Z}^s} d_j \phi(x-j) \right] \right| \leqslant \left| \sum_{j \in \mathbb{Z}^s} d_j \phi(-j) \right| + \sum_{k \in J} |t_k| \int_{\|x\| = r_k} dx \cdot \sup_{\|x\| \leqslant r_k} \left| \sum_{j \in \mathbb{Z}^s} d_j \phi(x-j) \right| \leqslant \hat{C} \sup_{x \in K} \left| \sum_{j \in \mathbb{Z}^s} d_j \phi(x-j) \right|, \tag{6}$$

where $\hat{C}$ is a positive constant. Now, (5) and (6) imply (1).

It remains to establish the identity (3). This is a matter of evaluating the integral of the power of a Euclidean distance function over a sphere and in order to do this we use the following approach. We observe that with $\eta := \|y\|$, the left-hand side of (3) is the same as

$$\sum_{l=0}^{(m-1)/2} a_l \int_{\|x\|=r} \|x-y\|^{m-2l}\, dx = H_s \sum_{l=0}^{(m-1)/2} a_l \eta^{m-2l} r^{s-1} (r/\eta)^{m-2l} \int_0^{\pi} (\sin \vartheta)^{s-2} \times \{1 + (\eta/r)^2 - 2(\eta/r)\cos\vartheta\}^{m/2-l}\, d\vartheta, \tag{7}$$

by changing from Cartesian to polar coordinates. Here, H_s is a positive constant. Making a change of variables and employing the generating function for Legendre polynomials P_j, which we can find in Rainville [3, p. 157], we conclude that the integral in (7) is, for $r < \|y\|$,

$$\int_{-1}^{1} (1-x^2)^{(s-3)/2} \{1 + (\eta/r)^2 - 2(\eta/r)x\}^{[m/2]-l+1-1/2}\, dx$$

$$= \sum_{j=0}^{\infty} \left(\frac{r}{\eta}\right)^{j+1} \int_{-1}^{1} (1-x^2)^{(s-3)/2} \times \{1 + (\eta/r)^2 - 2(\eta/r)x\}^{[m/2]-l+1} P_j(x)\, dx$$

$$= \sum_{j=0}^{s+[m/2]-l-2} \left(\frac{r}{\eta}\right)^{j+2l-m} \int_{-1}^{1} (1-x^2)^{(s-3)/2} \times \{1 + (r/\eta)^2 - 2(r/\eta)x\}^{[m/2]-l+1} P_j(x)\, dx,$$

where $[m/2]$ is the largest integer $\leqslant m/2$ and where we have made use of the orthogonality of the Legendre polynomials. By expanding the term in braces that occurs inside the integral and by making use of the orthogonality once more, we obtain that the above expression is the same as

$$\sum_{k=0}^{[m/2]-l+1} \binom{[m/2]-l+1}{k} \sum_{\substack{j=0 \\ j+l-[m/2]-k \text{ odd}}}^{s+[m/2]-l-2-k} \left(\frac{r}{\eta}\right)^{j+l-[m/2]-k}$$

$$\times (1+(r/\eta)^2)^k \int_{-1}^{1} (1-x^2)^{(s-3)/2} (-2x)^{[m/2]-l+1-k} P_j(x)\, dx. \quad (8)$$

Here, the requirement that $j+l-[m/2]-k$ be odd comes from the fact that symmetry implies that

$$\int_{-1}^{1} (1-x^2)^{(s-3)/2} (-2x)^{[m/2]-l+1-k} P_j(x)\, dx = 0$$

if $j+l-[m/2]-k$ is even. Hence the powers of r/η that occur on the right-hand side of (7) when the integral therein is evaluated, are all the even numbers from 0 up to $s+m-2l-2$. Thus, combining (7) and (8) gives (3), and we observe that

$$b_{l,0} \equiv b_0 = \int_{-1}^{1} (1-x^2)^{(s-3)/2}\, dx > 0.$$

The theorem is proved.

We remark that this theorem is a generalization of a theorem due to Powell (1989, private communication) that proves stability for $\phi(r)=r$ when $s=3$. Further, the authors thank A. Iserles for pointing out the usefulness of Legendre polynomials in the above computations. They also thank the two referees for several thoughtful comments.

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